

On Quantum Field Theory with Nonzero Minimal Uncertainties in Positions and Momenta

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Abstract

We continue studies on quantum field theories on noncommutative geometric spaces, focusing on classes of noncommutative geometries which imply ultraviolet and infrared modifications in the form of nonzero minimal uncertainties in positions and momenta. The case of the ultraviolet modified uncertainty relation which has appeared from string theory and quantum gravity is covered. The example of euclidean ϕ^4 -theory is studied in detail and in this example we can now show ultraviolet and infrared regularisation of all graphs.

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1 Introduction

There has been considerable progress in several branches of the mathematics of non-commutative or ‘quantum’ geometry which, in a broad sense, is the generalisation of geometric concepts and tools to situations in which the algebra of functions on a manifold becomes noncommutative. The physical motivations range e.g. from integrable models and generalised symmetry groups to studies on the algebraic structure of the Higgs sector in the standard model. Standard references are e.g. [1]-[9].

Here, we continue the approach of [10]-[16] in which is studied the quantum mechanics on certain ‘noncommutative geometries’ where

$$[\mathbf{x}_i, \mathbf{x}_j] \neq 0 \quad \text{and} \quad [\mathbf{p}_i, \mathbf{p}_j] \neq 0 \quad (1)$$

and in particular where:

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar(\delta_{ij} + \alpha_{ijkl}\mathbf{x}_k\mathbf{x}_l + \beta_{ijkl}\mathbf{p}_k\mathbf{p}_l + \dots) \quad (2)$$

A crucial feature of the generalised commutation relations, which we will discuss in Sec.2, is that for appropriate matrices $\alpha, \beta \in M_{n^4}(\mathbf{C})$ one finds ordinary quantum mechanical behaviour at medium scales, while as a new effect at very small and very large scales there appear nonzero minimal uncertainties $\Delta x_0, \Delta p_0$ in positions and in momenta.

The main part of the paper is Sec.3, where we proceed with the study of a previously suggested approach to the formulation of quantum field theories on such geometries. For the example of ϕ^4 -theory we can now explicitly show that minimal uncertainties in positions and momenta do have the power to regularise all graphs in the ultraviolet and the infrared.

The underlying motivation is the idea is that nonvanishing minimal uncertainties in positions and momenta could be effects caused by gravity, or string theory. The possible gravitational origins for modifications in the ultraviolet and in the infrared are to be considered separately:

On the one hand, in order to resolve small distances test particles need high energies. The latest at the Planck scale of about $10^{-35}m$ the gravity effects of high energetic test particles must significantly disturb the spacetime structure which was tried to be resolved. It has therefore long been suggested that there exists a finite limit to the possible resolution of distances. Probably the simplest ansatz for its quantum theoretical expression is that of a nonvanishing minimal uncertainty in positions. This ansatz covers an ultraviolet behaviour which has been found in string theory, as well as in quantum gravity, arising from an effective uncertainty relation:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \text{const} \cdot \Delta p \quad (3)$$

References are e.g. [17]-[22]; a recent review is [24].

On the other hand, minimal uncertainties in momentum, as an infrared effect, may arise from large scale gravity. The argument is related to the fact that on a general curved spacetime there is no notion of a plane wave, i.e. of exact localisation in momentum space, see [14, 16].

We remark that in the case of minimal uncertainties in positions only, examples are known of noncommutative geometries of the type of Eqs.1,2 which preserve the Poincaré symmetry, i.e. where the universal enveloping algebra of the Poincaré Lie algebra is a $*$ -sub algebra of the Heisenberg algebra, see [25, 26]. Generally however, we take the view that similarly to curved spaces which may preserve some of the flat space symmetries while breaking others, also noncommutative geometric spaces, as defined through commutation relations, may preserve some symmetries while breaking others.

Here, we therefore study the general case, i.e. not assuming a specific symmetry, and allowing the existence both of minimal uncertainties in positions and in momenta.

An alternative approach with a similar motivation, but based on the canonical formulation of quantum field theory, is [27]. Other approaches to nonrelativistic quantum mechanics with generalised commutation relations, mostly motivated by quantum groups, and related studies, are e.g. [28]-[43].

2 Quantum mechanics with nonzero minimal uncertainties

2.1 Uncertainty relations

We review and generalise the results of [10]-[13] on nonrelativistic quantum mechanics with nonzero minimal uncertainties in positions and momenta.

Let \mathcal{A} denote the associative Heisenberg algebra generated by elements $\mathbf{x}_i, \mathbf{p}_j$ that obey generalised commutation relations of the form of Eqs.1,2. The modified commutation relations are required to be consistent with the $*$ -involution $\mathbf{x}^* = \mathbf{x}, \mathbf{p}^* = \mathbf{p}$, implying that α and β obey $\alpha_{ijkl}^* = \alpha_{ijlk}, \beta_{ijkl}^* = \beta_{ijlk}$.

The study of the uncertainty relations that belong to the Heisenberg algebra \mathcal{A} yields information that holds independently of the choice of representation. Let us therefore assume the $\mathbf{x}_i, \mathbf{p}_j$ to be represented as symmetric operators obeying the new commutation relations on some dense domain $D \subset \mathcal{H}$ in a Hilbert space \mathcal{H} . On this space D of physical states one derives uncertainty relations of the form

$$\Delta A \Delta B \geq 1/2 |\langle [A, B] \rangle| \quad (4)$$

so that e.g. $[\mathbf{x}_i, \mathbf{x}_j] \neq 0$, yields $\Delta x_i \Delta x_j \geq 0$. Their noncommutativity implies that the \mathbf{x}_i (as well as the \mathbf{p}_i) can no longer be simultaneously diagonalised. Because of the modified commutation relations Eqs.2 and the corresponding uncertainty relations

there can appear the even more drastic effect that the \mathbf{x}_i (as well as the \mathbf{p}_j) may also not be diagonalisable separately. Instead there then exist nonzero minimal uncertainties in positions and momenta. The mechanism can be seen also in one dimension, to which case we will restrict ourselves until Sec.2.5. We consider Eq.2 with $\alpha, \beta > 0$ and $\alpha\beta < 1/\hbar^2$:

$$[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \alpha\mathbf{x}^2 + \beta\mathbf{p}^2) \quad (5)$$

For fixed but sufficiently small α and β one finds ordinary quantum mechanical behaviour at medium scales while e.g. the term proportional to β contributes for large $\langle \mathbf{p}^2 \rangle = \langle \mathbf{p} \rangle^2 + (\Delta p)^2$ i.e. in the ultraviolet. Similarly the term proportional to α leads to an infrared effect. The uncertainty relation to Eq.5 is:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha(\Delta x)^2 + \alpha\langle \mathbf{x} \rangle^2 + \beta(\Delta p)^2 + \beta\langle \mathbf{p} \rangle^2 \right) \quad (6)$$

It implies nonzero minimal uncertainties in \mathbf{x} - as well as in \mathbf{p} - measurements. This can be seen as follows: As e.g. Δx gets smaller, Δp must increase so that the product $\Delta x \Delta p$ of the LHS remains larger than the RHS. In usual quantum mechanics this is always possible, i.e. Δx can be made arbitrarily small. However, in the generalised case, for $\alpha, \beta > 0$ there is a positive $(\Delta p)^2$ term on the RHS which eventually grows faster with Δp than the LHS. Thus Δx can no longer become arbitrarily small. The minimal uncertainty in \mathbf{x} depends on the expectation value in position and momentum via

$$k := \alpha\langle \mathbf{x} \rangle^2 + \beta\langle \mathbf{p} \rangle^2 \quad (7)$$

and is explicitly:

$$\Delta x_0 = \sqrt{\frac{(1+k)\beta\hbar^2}{1-\alpha\beta\hbar^2}} \quad (8)$$

Analogously one obtains the smallest uncertainty in momentum

$$\Delta p_0 = \sqrt{\frac{(1+k)\alpha\hbar^2}{1-\alpha\beta\hbar^2}} \quad (9)$$

with the absolutely smallest uncertainties obtained for $k = 0$.

Note that if there was e.g. an \mathbf{x} - eigenstate $|\psi\rangle \in D$ with $\mathbf{x} \cdot |\psi\rangle = \lambda|\psi\rangle$ it would have no uncertainty in position (we always assume states $|\psi\rangle$ to be normalised):

$$(\Delta x)_{|\psi\rangle}^2 = \langle \psi | (\mathbf{x} - \langle \psi | \mathbf{x} | \psi \rangle)^2 | \psi \rangle = 0 \quad (10)$$

which would be a contradiction. There are thus no physical states $|\psi\rangle \in D$ which are eigenstates of \mathbf{x} or \mathbf{p} .

Thus, for any physical domain D , i.e. for all $*$ -representations of the commutation relations, there are no physical states in the ‘minimal uncertainty gap’:

$$\nexists \quad |\psi\rangle \in D : \quad 0 \leq (\Delta x)_{|\psi\rangle} < \Delta x_0 \quad (11)$$

$$\nexists \quad |\psi\rangle \in D : \quad 0 \leq (\Delta p)_{|\psi\rangle} < \Delta p_0 \quad (12)$$

Crucially, unlike on ordinary geometry, there do not exist sequences $\{|\psi_n\rangle\}$ of physical states which would approximate point localisations in position or momentum space, i.e. for which the uncertainty would decrease to zero:

$$\nexists \quad |\psi_n\rangle \in D : \quad \lim_{n \rightarrow \infty} (\Delta x)_{|\psi_n\rangle} = 0. \quad \text{or} \quad \lim_{n \rightarrow \infty} (\Delta p)_{|\psi_n\rangle} = 0. \quad (13)$$

Heisenberg algebras \mathcal{A} with these generalised canonical commutation relations therefore no longer have spectral representations on wave functions $\langle x|\psi\rangle$ or $\langle p|\psi\rangle$.

2.2 Bargmann Fock representation

For practical calculations and for detailed studies of the functional analysis a Hilbert space representation of the generalised Heisenberg algebra is needed. We generalise the Bargmann Fock representation.

In ordinary quantum mechanics the Bargmann Fock representation is unitarily equivalent to the position and the momentum representation, being the spectral representation of the operator $\bar{\eta}\partial_{\bar{\eta}} \in \mathcal{A}$ where:

$$\bar{\eta} := \frac{1}{2L}\mathbf{x} - \frac{i}{2K}\mathbf{p} \quad \text{and} \quad \partial_{\bar{\eta}} := \frac{1}{2L}\mathbf{x} + \frac{i}{2K}\mathbf{p} \quad (14)$$

Here L and K are length and momentum scales, related by $LK = \hbar/2$. Thus $\bar{\eta}$ and $\partial_{\bar{\eta}}$ obey $\partial_{\bar{\eta}}\bar{\eta} - \bar{\eta}\partial_{\bar{\eta}} = 1$, which is of the form of a Leibniz rule and justifies the notation. One readily finds the countable set of eigenvectors $\bar{\eta}\partial_{\bar{\eta}}|\bar{\eta}^n\rangle = n|\bar{\eta}^n\rangle$ with $n = 0, 1, 2, \dots$. With the definitions $|a\bar{\eta}^n + b\bar{\eta}^m\rangle := |a\bar{\eta}^n\rangle + |b\bar{\eta}^m\rangle$ and $a|\bar{\eta}^n\rangle := |a\bar{\eta}^n\rangle$ arbitrary states $|\psi\rangle$ are written as polynomials or power series

$$|\psi\rangle = \left| \sum_{r=0}^{\infty} \psi_r \frac{\bar{\eta}^r}{\sqrt{r!}} \right\rangle = |\psi(\bar{\eta})\rangle \quad (15)$$

on which \mathbf{x} and \mathbf{p} are represented in terms of multiplication and differentiation operators

$$\mathbf{x} = L(\bar{\eta} + \partial_{\bar{\eta}}) \quad \mathbf{p} = iK(\bar{\eta} - \partial_{\bar{\eta}}) \quad (16)$$

The well known formula for the scalar product of states is

$$\langle \psi|\phi\rangle = \frac{1}{2\pi i} \int d\eta d\bar{\eta} \, \overline{\psi(\bar{\eta})} \, e^{-\bar{\eta}\eta} \, \phi(\bar{\eta}) \quad (17)$$

Here the $\psi(\bar{\eta})$ and $\phi(\bar{\eta})$ on the RHS are to be read as polynomials or power series in ordinary complex variables rather than as elements of \mathcal{A} .

A key observation for the generalisation of the Bargmann Fock representation is that the scalar product can be expressed without relying to complex integration [10]:

$$\langle \psi|\phi\rangle = \overline{\psi(\bar{\eta})} \, e^{\partial_{\eta}\partial_{\bar{\eta}}} \, \phi(\bar{\eta}) \, |_{\eta=0=\bar{\eta}} \quad (18)$$

The exponential is defined through its power series i.e. $e^{\partial_\eta \partial_{\bar{\eta}}} = \sum_{r=0}^{\infty} \frac{\partial_\eta \partial_{\bar{\eta}}^r}{r!}$ where the derivatives $\partial_{\bar{\eta}}$ act from the left while the derivatives ∂_η act from the right. The evaluation procedure is to carry out the differentiations and then to set η and $\bar{\eta}$ equal to zero. The remaining number is the value of the scalar product. This can be done purely algebraically by using the Leibniz rule $\partial_{\bar{\eta}} \bar{\eta} - \bar{\eta} \partial_{\bar{\eta}} = 1$ and its complex conjugate $\eta \partial_\eta - \partial_\eta \eta = 1$. For example

$$\begin{aligned} \partial_{\bar{\eta}} \bar{\eta}^2 &= \partial_{\bar{\eta}} \bar{\eta} \bar{\eta} = (\bar{\eta} \partial_{\bar{\eta}} + 1) \bar{\eta} = \bar{\eta} \partial_{\bar{\eta}} \bar{\eta} + \bar{\eta} \\ &= \bar{\eta} (\bar{\eta} \partial_{\bar{\eta}} + 1) + \bar{\eta} = \bar{\eta} \bar{\eta} \partial_{\bar{\eta}} + \bar{\eta} + \bar{\eta} = 2\bar{\eta} \end{aligned}$$

and

$$\eta^2 \partial_\eta = \eta (\partial_\eta \eta + 1) = \dots = 2\eta$$

Thus e.g.:

$$\begin{aligned} \langle \bar{\eta}^2 | 2 + 3\bar{\eta}^2 \rangle &= \eta^2 e^{\partial_\eta \partial_{\bar{\eta}}} (2 + 3\bar{\eta}^2) |_{\eta=0=\bar{\eta}} \\ &= \eta^2 \sum_{r=0}^{\infty} \frac{\partial_\eta \partial_{\bar{\eta}}^r}{r!} (2 + 3\bar{\eta}^2) |_{\eta=0=\bar{\eta}} \\ &= 3\eta^2 \frac{\partial_\eta^2 \partial_{\bar{\eta}}^2}{2} \bar{\eta}^2 |_{\eta=0=\bar{\eta}} = 6 \end{aligned}$$

Since the scalar product formula Eq.17 relies on conventional commutative integration over the complex plane, it cannot be used in the generalised case where e.g. in n dimensions the $\bar{\eta}_i$ will be noncommutative. It is however possible to use a generalisation (Eq.24) of Eq.18 (which can also be applied in the fermionic case instead of using Berezin integration [10]). Also in one dimension it allows to construct a Bargmann Fock Hilbert space representation for Eq.5.

To this end we rewrite Eq.5 in the form

$$[\mathbf{x}, \mathbf{p}] = i\hbar + i\hbar(q^2 - 1) \left(\frac{\mathbf{x}^2}{4L^2} + \frac{\mathbf{p}^2}{4K^2} \right) \quad (19)$$

where the parameter $q \geq 1$ measures the deviation from the ordinary commutation relations. The length and momentum scales are related by $LK = \hbar(q^2 + 1)/4$. We can now again represent \mathbf{x} and \mathbf{p} as the usual linear combinations (Eq.16) of generators $\bar{\eta}$ and $\partial_{\bar{\eta}}$. A complete generalised Bargmann Fock calculus is defined as the complex associative algebra \mathcal{B} with the commutation relations

$$\partial_{\bar{\eta}} \bar{\eta} - q^2 \bar{\eta} \partial_{\bar{\eta}} = 1 \quad \eta \partial_\eta - q^2 \partial_\eta \eta = 1 \quad (20)$$

$$\bar{\eta} \partial_\eta - q^2 \partial_\eta \bar{\eta} = 0 \quad \partial_{\bar{\eta}} \eta - q^2 \eta \partial_{\bar{\eta}} = 0 \quad (21)$$

$$\eta \bar{\eta} - q^2 \bar{\eta} \eta = 0 \quad \partial_\eta \partial_{\bar{\eta}} - q^2 \partial_{\bar{\eta}} \partial_\eta = 0 \quad (22)$$

A short calculation shows that the commutation relation Eq.19 in fact uniquely translates into the commutation relations Eqs.20 through Eq.16, see [13]. On the other

hand, the commutation relations Eqs.21,22 are nonunique and could also be chosen commutative. Our choice is the special case of the choice made for the n dimensional case in [10] under the requirements of a quantum group module algebra structure, invariance of the Poincaré series and simple form of the scalar product formula. These requirements are here not physically relevant, but it is convenient to use the formulas already obtained for this case. Generally, other choices for the commutation relations between the barred and the unbarred generators are possible and lead to respectively more or less simple to evaluate formulations of the scalar product. These representation specific choices do of course not affect the physical content of the theory, such as the uncertainty relations, transition amplitudes or expectation values.

The Heisenberg algebra \mathcal{A} is now represented on the domain D of polynomials in $\bar{\eta}$

$$D := \{|\psi\rangle \mid \psi(\bar{\eta}) = \text{polynomial}(\bar{\eta})\} \quad (23)$$

with the action of \mathbf{x} and \mathbf{p} given by Eq.16 where the differentiations are to be evaluated algebraically using the generalised Leibniz rule given in Eqs.20. As is the case on ordinary geometry, the operators $\bar{\eta}$ and $\partial_{\bar{\eta}}$ are mutually adjoint with respect to the unique and positive definite scalar product, which now takes the form:

$$\langle \psi | \phi \rangle = \overline{\psi(\bar{\eta})} e_{1/q}^{\partial_{\eta} \partial_{\bar{\eta}}} \phi(\bar{\eta}) \mid_{\eta=0=\bar{\eta}} \quad (24)$$

The q - exponential is defined through

$$e_{1/q}^{\partial_{\eta} \partial_{\bar{\eta}}} = \sum_{r=0}^{\infty} \frac{(\partial_{\eta_i} \partial_{\bar{\eta}_i})^r}{[r]_{1/q}!} \quad (25)$$

where the derivatives ∂_{η} act from the right and where

$$[r]_c := 1 + c^2 + c^4 + \dots + c^{2(r-1)} = \frac{c^{2r} - 1}{c^2 - 1}$$

and

$$[r]_c! := 1 \cdot [2]_c \cdot [3]_c \cdot \dots \cdot [r]_c$$

The evaluation procedure is again to algebraically carry out the differentiations, now using Eqs.20-22 and then to set η and $\bar{\eta}$ equal to zero. The remaining number is the value of the scalar product.

The functional analysis of the position and momentum operators is as follows: We denote by H the Hilbert space obtained by completion with respect to the norm induced by the scalar product. A Hilbert basis is given by the orthonormal family

$$\left\{ ([r]_q!)^{-1/2} |\bar{\eta}^r\rangle \mid r = 0, 1, 2, \dots \right\} \quad (26)$$

The domain $D \subset H$, which is dense in H , is a physical domain, i.e. on it the \mathbf{x} and \mathbf{p} are represented as symmetric operators obeying the commutation relation Eq.19.

In fact D is also analytic since $\mathbf{x}.D \subset D$ and $\mathbf{p}.D \subset D$, i.e. D is a $*$ - \mathcal{A} module. The \mathbf{x} and \mathbf{p} are no longer essentially self-adjoint. Their adjoints \mathbf{x}^* and \mathbf{p}^* are closed but nonsymmetric. The \mathbf{x}^{**} and \mathbf{p}^{**} are closed and symmetric. Their deficiency subspaces are of finite (nonzero) and equal dimension so that there are continuous families of self adjoint extensions in H . Crucially however, because of the minimal uncertainties in positions and momenta, neither \mathbf{x} nor \mathbf{p} have self-adjoint extensions neither in D nor in any other physical domain, i.e. not in any other $*$ -representation of the commutation relations. For the details and proofs see [13].

One arrives at the following picture:

While in classical mechanics the states can have exact positions and momenta, in quantum mechanics there is the uncertainty relation that does not allow \mathbf{x} and \mathbf{p} to have common eigenvectors. Nevertheless \mathbf{x} and \mathbf{p} separately do have 'eigenvectors', though non-normalisable ones. The spectrum is continuous, namely the configuration or momentum space. The position and momentum operators are essentially self-adjoint. Our generalisation of the Heisenberg algebra has further consequences for the observables \mathbf{x} and \mathbf{p} : It is not only that the \mathbf{x} and \mathbf{p} have no common eigenstates. The uncertainty relation now implies that they do not have any eigenvectors in the representation of the Heisenberg algebra. Although \mathbf{x} and \mathbf{p} separately do have self-adjoint extensions, they do not have self-adjoint extensions on any physical domain i.e. not on any $*$ -representation of both \mathbf{x} and \mathbf{p} . This means the non-existence of absolute precision in position or momentum measurements. Instead there are absolutely minimal uncertainties in these measurements which are, in terms of the new variables of Eq.19:

$$\Delta x_0 = L\sqrt{1 - q^{-2}}, \quad \Delta p_0 = K\sqrt{1 - q^{-2}} \quad (27)$$

Recall that due to Eq.10 the non self-adjointness and non-diagonalisability of \mathbf{x} and \mathbf{p} is necessary to allow for the physical description of minimal uncertainties. Note that on the other hand the fact that \mathbf{x} and \mathbf{p} still have the slightly weaker property of being symmetric is sufficient to guarantee that all physical expectation values are real.

2.3 Maximal localisation states

Generally, all information on positions and momenta is encoded in the matrix elements of the position and momentum operators, and matrix elements can of course be calculated in any basis. In the Bargmann Fock basis matrix elements e.g. of the position operators are calculated as

$$\langle \psi | \mathbf{x} | \phi \rangle = \overline{\psi(\bar{\eta})} e_{1/q}^{\partial_{\eta} \partial_{\bar{\eta}}} L(\bar{\eta} + \partial_{\bar{\eta}}) \phi(\bar{\eta}) |_0 \quad (28)$$

Ordinarily, information on position or momentum can conveniently be obtained by projection onto position or momentum eigenstates $\langle x | \psi \rangle$ or $\langle p | \psi \rangle$ i.e. by using a

position or momentum representation.

That there are now no more physical \mathbf{x} - or \mathbf{p} - eigenstates, can also be seen directly in the Bargmann Fock representation. We consider e.g. the eigenvalue problem for \mathbf{x}

$$\mathbf{x} \cdot |\psi_\lambda\rangle = \lambda |\psi_\lambda\rangle \quad \text{i.e.} \quad L(\bar{\eta} + \partial_{\bar{\eta}}) \psi_\lambda(\bar{\eta}) = \lambda \psi_\lambda(\bar{\eta}) \quad (29)$$

which yields a recursion formula for the coefficients of the expansion:

$$\psi_\lambda(\bar{\eta}) = \sum_{r=0}^{\infty} \psi_{\lambda,r} \bar{\eta}^r \quad (30)$$

In ordinary quantum mechanics the solution is a Dirac δ 'function', transformed into Bargmann Fock space, (i.e. Eq.102 with λ instead of x_0). In the generalised setting it is interesting to see the effect of the appearance of the minimal uncertainty 'gap'.

The (no longer generally mutually orthogonal) solutions $\sum_{r=0}^{\infty} \psi_{\lambda,r} \bar{\eta}^r$ to Eq.29 have vanishing uncertainty in positions but they are not contained in the domain of \mathbf{p} (this would of course contradict the uncertainty relation) and they are therefore not physical states. However every polynomial approximation to the power series is contained in the physical domain D , i.e. $\sum_{r=0}^n \psi_{\lambda,r} \bar{\eta}^r \in D$ for arbitrary finite n . Thus, each $\sum_{r=0}^n \psi_{\lambda,r} \bar{\eta}^r$ has an \mathbf{x} - uncertainty which is in fact larger than Δx_0 . For details and a graph of their scalar product see [13].

Let us now consider the physical states $|\phi_{\xi,\pi}^{mlx}\rangle, |\phi_{\xi,\pi}^{mlp}\rangle$ which have the maximal localisation in \mathbf{x} or \mathbf{p} for given expectation values ξ, π in positions and momenta:

$$\Delta x_{|\phi_{\xi,\pi}^{mlx}\rangle} = \Delta x_0 \quad (31)$$

$$\langle \phi_{\xi,\pi}^{mlx} | \mathbf{x} | \phi_{\xi,\pi}^{mlx} \rangle = \xi, \quad \langle \phi_{\xi,\pi}^{mlp} | \mathbf{p} | \phi_{\xi,\pi}^{mlp} \rangle = \pi \quad (32)$$

with Δx_0 given by Eq.8 and similarly for $|\phi_{\xi,\pi}^{mlp}\rangle$. E.g. the projection $\langle \phi_{\xi,\pi}^{mlx} | \psi \rangle$ is then the probability amplitude for finding the particle maximally localised around ξ with momentum expectation π . For $\alpha, \beta \rightarrow 0$ one recovers the position and the momentum eigenvectors.

In order to calculate e.g. the $|\phi_{\xi,\pi}^{mlx}\rangle$ we use that these physical states realise the equality in the uncertainty relation. As is well known the uncertainty relation follows from the positivity of the norm:

$$|(\mathbf{x} - \langle \mathbf{x} \rangle + \frac{\langle [\mathbf{x}, \mathbf{p}] \rangle}{2(\Delta p)^2} (\mathbf{p} - \langle \mathbf{p} \rangle)) | \psi \rangle| \geq 0 \quad (33)$$

which is

$$\langle \psi | (\mathbf{x} - \langle \mathbf{x} \rangle)^2 - \left(\frac{|\langle [\mathbf{x}, \mathbf{p}] \rangle|}{2(\Delta p)^2} \right)^2 (\mathbf{p} - \langle \mathbf{p} \rangle)^2 | \psi \rangle \geq 0 \quad (34)$$

so that:

$$\Delta x \Delta p \geq \frac{|\langle [\mathbf{x}, \mathbf{p}] \rangle|}{2} \quad (35)$$

Thus, a state $|\psi\rangle$ obeys $\Delta x \Delta p = |\langle[\mathbf{x}, \mathbf{p}]\rangle|/2$, i.e. it is on the boundary of the physically allowed region if:

$$(\mathbf{x} - \langle\mathbf{x}\rangle + \frac{\langle[\mathbf{x}, \mathbf{p}]\rangle}{2(\Delta p)^2}(\mathbf{p} - \langle\mathbf{p}\rangle))|\psi\rangle = 0 \quad (36)$$

In any given representation this equation has a family of squeezed state solutions parametrized by $\langle\mathbf{x}\rangle, \langle\mathbf{p}\rangle, \Delta x, \Delta p$ where the four parameters obey Eq.6 with the equality sign. Choosing for Δx or Δp the minimal values given by Eqs.8,9 yields the maximal localisation states $|\phi_{\xi,\pi}^{mlx}\rangle$ and $|\phi_{\xi,\pi}^{mlp}\rangle$.

In [25], we calculated maximal localisation states in the case $\alpha = 0$. The absence of a minimal uncertainty in momentum there allows a spectral representation of \mathbf{p} , with Eq.36 taking the form of an exactly solvable differential equation. In particular, the new concept of quasi-position representation has been introduced, where the Heisenberg algebra is represented on the wave functions $\psi(\xi) := \langle\phi_{\xi,0}^{mlx}|\psi\rangle$. Related to the minimal uncertainty in positions there appears a minimal wavelength in quasi-position space.

In the general situation with minimal uncertainties in positions and in momenta we work in Bargmann Fock space where Eq.36 takes the form (using Eq.7)

$$\left(L(\bar{\eta} + \partial_{\bar{\eta}}) - \langle\mathbf{x}\rangle + i\hbar \frac{1 + \alpha(\Delta x)^2 + \beta(\Delta p)^2 + k}{2(\Delta p)^2} (iK(\bar{\eta} - \partial_{\bar{\eta}}) - \langle\mathbf{p}\rangle) \right) \psi(\bar{\eta}) = 0 \quad (37)$$

yielding a three terms recurrence relation for the coefficients of the expansion of $|\psi\rangle$ in $\bar{\eta}$. The solutions, i.e. the maximal localisation states, can be expressed in terms of so-called q -continuous Hermite functions. A detailed study of the maximal localisation states and the corresponding quasi-position and quasi-momentum representation has been carried out in [44]. A survey of q -special functions is [45].

A further problem is to find a generalised Fourier transformation that allows to easily transform information on positions into information on momenta. While this has been worked out for the special case $\alpha = 0$ in [25], here the recent work [46] may be relevant. In this context, compare also with the generalised quantum mechanics (with discrete x - and p -spectra) developed in [39, 40], where techniques developed in [41] lead to generalised Fourier transformations.

2.4 Integral kernels and Green functions

Elements $P = P(\mathbf{x}, \mathbf{p}) \in \mathcal{A}$ of the Heisenberg algebra do not only have representations in terms of Bargmann Fock operators $P(\bar{\eta}, \partial_{\bar{\eta}})$, via Eq.16 but can also still be represented as integral kernels. Once the operator $P(\bar{\eta}, \partial_{\bar{\eta}})$ is normal ordered, there is a simple rule for deriving its integral kernel G_P , which is a function of $\bar{\eta}'$ and η . Integrating any Bargmann Fock function $\psi(\bar{\eta})$ over $G_P(\bar{\eta}', \eta)$ leads then to a function of $\bar{\eta}'$, which is $P.\psi(\bar{\eta}')$. Generalising

$$P(\bar{\eta}', \partial_{\bar{\eta}'}). \psi(\bar{\eta}') = \frac{1}{2\pi i} \int d\bar{\eta} d\eta G_P(\bar{\eta}', \eta) e^{-\bar{\eta}\eta} \psi(\bar{\eta}) \quad (38)$$

one now has

$$P(\bar{\eta}', \partial_{\bar{\eta}'}).\psi(\bar{\eta}') = \int d\bar{\eta}d\eta G_P(\bar{\eta}', \eta) e_{1/q}^{\partial_{\bar{\eta}}\partial_{\eta}} \psi(\bar{\eta}) \quad (39)$$

Here the integration is meant to be the algebraic scalar product which expresses the integration in terms of derivatives, i.e. one defines:

$$\int d\bar{\eta}d\eta \overline{\psi(\bar{\eta})} e_{1/q}^{\partial_{\bar{\eta}}\partial_{\eta}} \phi(\bar{\eta}) := \overline{\psi(\bar{\eta})} e_{1/q}^{\partial_{\bar{\eta}}\partial_{\eta}} \phi(\bar{\eta}) |_{\eta=0=\bar{\eta}} \quad (40)$$

For this to work, the appropriate commutation relations between two copies (e.g. primed and unprimed) of the function space had to be calculated, see [11].

E.g. the position operator $\mathbf{x} := L(\bar{\eta} + \partial_{\bar{\eta}})$, has the integral kernel

$$G_x(\bar{\eta}', \eta) = L (\bar{\eta}' e_{1/q}^{\bar{\eta}'\eta} + e_{1/q}^{\bar{\eta}'\eta} \eta) \quad (41)$$

Another example is the harmonic oscillator $H := \omega \bar{\eta} \partial_{\bar{\eta}}$. Since H is self-adjoint, the time evolution operator $U = e^{-i(t_f - t_i)H}$ is unitary. The eigenvalues of H are:

$$H |\bar{\eta}^r\rangle = \omega [r]_q |\bar{\eta}^r\rangle \quad (42)$$

The integral kernel of U , i.e. the Greens function is then found to be [11]:

$$G_U = \sum_{r=0}^{\infty} \frac{(\bar{\eta}'\eta)^r}{[r]_{1/q}!} e^{-i\omega(t_f - t_i)[r]_q} \quad (43)$$

reducing for $q \rightarrow 1$ to the well known result:

$$G_U(\bar{\eta}', \eta) = e^{\bar{\eta}'\eta e^{-i\omega(t_f - t_i)}} \quad (44)$$

2.5 n - dimensional generalisations

Let us come back to the full n - dimensional situation with commutation relations of the form of Eqs.1,2. Obviously, terms $\alpha_{ijii} > 0$ and $\beta_{ijii} > 0$ are sufficient to induce minimal uncertainties in momenta and positions, thus excluding spectral representations of the \mathbf{x}_i or \mathbf{p}_j , and therefore complicating the construction of Hilbert space representations of the Heisenberg algebra.

There are however n - dimensional generalisations of our q - Bargmann Fock space which straightforwardly supply Hilbert space representations for certain classes of generalised Heisenberg algebras. We will use two of them as examples of fixed ‘background’ geometries.

The first example is the Heisenberg algebra \mathcal{A}_1 , defined as the tensor product of n commuting copies of the one-dimensional algebra \mathcal{A} (all $q_i \geq 1$):

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij} + i\hbar\delta_{ij}(q_i^2 - 1) \left(\frac{1}{4L_i^2} \mathbf{x}_i^2 + \frac{1}{4K_i^2} \mathbf{p}_i^2 \right) \quad (45)$$

$$[\mathbf{x}_i, \mathbf{x}_j] = 0, \quad [\mathbf{p}_i, \mathbf{p}_j] = 0 \quad (46)$$

where

$$L_i K_i = \hbar(q_i^2 + 1)/4 \quad (47)$$

The Heisenberg algebra \mathcal{A}_1 has an obvious Hilbert space representation on the domain $D_1 \subset H_1$ which is the n fold tensor product of the previously considered domains D in the Hilbert space H_1 , spanned by the orthogonal polynomials $\bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdots \bar{\eta}_n^{r_n}$, with norm:

$$\langle \bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdots \bar{\eta}_n^{r_n} | \bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdots \bar{\eta}_n^{r_n} \rangle = \prod_{i=1}^n [r_i]_{q_i}! \quad (48)$$

As the second example, now with nontrivial commutation relations also among the \mathbf{x}_i and among the \mathbf{p}_i we consider the Heisenberg algebra \mathcal{A}_2 defined through:

$$[\mathbf{x}_r, \mathbf{p}_r] = i\hbar + i\hbar(q^2 - 1) \sum_{s \leq r} \left(\frac{q^2 + 1}{2} \right)^{s-1} \left(\frac{\mathbf{x}_s^2}{4L_s^2} + \frac{\mathbf{p}_s^2}{4K_s^2} \right) \quad (49)$$

and mixed commutation relations for $s > r$

$$[\mathbf{x}_s, \mathbf{p}_r] = -i \frac{K_r}{L_r} \frac{q-1}{q+1} \{\mathbf{x}_s, \mathbf{x}_r\} \quad [\mathbf{x}_s, \mathbf{x}_r] = -i \frac{L_r}{K_r} \frac{q-1}{q+1} \{\mathbf{x}_s, \mathbf{p}_r\} \quad (50)$$

and for $s < r$

$$[\mathbf{x}_s, \mathbf{p}_r] = i \frac{L_s}{K_s} \frac{q-1}{q+1} \{\mathbf{p}_s, \mathbf{p}_r\} \quad [\mathbf{p}_s, \mathbf{p}_r] = -i \frac{K_s}{L_s} \frac{q-1}{q+1} \{\mathbf{x}_s, \mathbf{p}_r\} \quad (51)$$

with:

$$L_r K_r := \frac{\hbar}{2} \left(\frac{q^2 + 1}{2} \right)^r \quad (52)$$

In order to represent \mathcal{A}_2 we define the generalised Bargmann Fock calculus as the complex algebra \mathcal{B}_2 with commutation relations (the i, j summed over):

$$\bar{\eta}_a \bar{\eta}_b - \frac{1}{q} R_{ba}^{ji} \bar{\eta}_j \bar{\eta}_i = 0 \quad (53)$$

$$\partial_{\bar{\eta}^a} \bar{\eta}_b - q R_{ib}^{aj} \bar{\eta}_j \partial_{\bar{\eta}^i} = \delta_{ab} \quad \partial_{\bar{\eta}^a} \partial_{\bar{\eta}^b} - \frac{1}{q} R_{ab}^{ij} \partial_{\bar{\eta}^j} \partial_{\bar{\eta}^i} = 0 \quad (54)$$

$$\partial_{\bar{\eta}^a} \partial_{\eta^b} - \frac{1}{q} (R^{-1})_{bi}^{ja} \partial_{\eta^j} \partial_{\bar{\eta}^i} = 0 \quad \partial_{\bar{\eta}^a} \eta_b - q R_{ab}^{ij} \eta_j \partial_{\bar{\eta}^i} = 0 \quad (55)$$

and their complex conjugates¹ where (the e_i^j are matrix units):

$$R = q \sum_i e_i^i \otimes e_i^i + \sum_{i \neq j} e_i^i \otimes e_j^j + (q - 1/q) \sum_{i > j} e_j^i \otimes e_i^j \quad (56)$$

¹Note that $\bar{\cdot}$ is an anti algebra morphism, so that e.g. $\overline{\partial_{\bar{\eta}^i} \bar{\eta}_j} = \eta_j \partial_{\eta^i}$ (we defined the ∂_{η^i} 's as right derivatives)

We can then represent \mathcal{A}_2 through

$$\mathbf{x}_r = L_r(\bar{\eta}_r + \partial_{\bar{\eta}^r}) \quad \text{and} \quad \mathbf{p}_r = iK_r(\bar{\eta}_r - \partial_{\bar{\eta}^r}) \quad (57)$$

on the domain of polynomials

$$D_2 := \{|\psi\rangle \mid \psi(\bar{\eta}) = \text{polynomial}(\bar{\eta}_1, \bar{\eta}_2, \dots, \bar{\eta}_n)\} \quad (58)$$

with the unique and positive definite scalar product

$$\langle \psi | \phi \rangle = \overline{\psi(\bar{\eta})} e^{\frac{\partial_{\eta_i} \partial_{\bar{\eta}_i}}{1/q}} \phi(\bar{\eta}) \big|_{\eta=0=\bar{\eta}} \quad (59)$$

which is a generalisation of Eq.18. The Hilbert space H_2 , completed with respect to the induced norm, has a Hilbert basis given by the orthogonal ordered polynomials $\bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdot \dots \cdot \bar{\eta}_n^{r_n}$ with norm:

$$\langle \bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdot \dots \cdot \bar{\eta}_n^{r_n} | \bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \cdot \dots \cdot \bar{\eta}_n^{r_n} \rangle = \prod_{i=1}^n [r_i]_q! \quad (60)$$

The Heisenberg algebra \mathcal{A}_2 and its Hilbert space representation has naturally appeared in the context of quantum groups [10]-[13] as a minimal generalisation under certain consistency conditions such as the invariance of the \ast - structure, Poincaré series, and the positivity of the norm. R in Eq.56 is the fundamental representation of the universal R-matrix that determines the quasitriangular structure of the quantum group $SU_q(n)$, which acts on \mathcal{A}_2 as linear quantum canonical transformations, i.e. \mathcal{A}_2 is a $SU_q(n)$ - \ast - comodule algebra.

Generally, a Hilbert space representation of fixed generalised commutation relations induces Hilbert space representations of a class of generalised commutation relations, simply by applying algebra isomorphisms ($M \in GL(n, \mathbf{R})$):

$$\mathbf{x}_r \rightarrow \mathbf{x}'_r = M_{rs}^{-1} \mathbf{x}_s \quad \mathbf{p}_r \rightarrow \mathbf{p}'_r = M_{sr} \mathbf{p}_s \quad (61)$$

E.g. the noncommutative geometries $\mathcal{A}_1, \mathcal{A}_2$ defined through Eqs.45-47 and Eqs.49-51 are of the form (summing over repeated indices):

$$[\mathbf{x}_r, \mathbf{p}_s] = i\hbar \delta_{rs} + i\hbar \alpha_{rstu} \{\mathbf{x}_t, \mathbf{x}_u\} + i\hbar \beta_{rstu} \{\mathbf{p}_t, \mathbf{p}_u\} \quad (62)$$

$$[\mathbf{x}_r, \mathbf{x}_s] = i\mu_{rstu} \{\mathbf{x}_t, \mathbf{p}_u\} \quad (63)$$

$$[\mathbf{p}_r, \mathbf{p}_s] = i\nu_{rs,tu} \{\mathbf{x}_t, \mathbf{p}_u\} \quad (64)$$

with the α, β, μ, ν real matrices. Through Eqs.61 one represents commutation relations of the same form Eqs.62-64 but specified through matrices $\alpha', \beta', \mu', \nu'$, where

$$\alpha'_{abcd} = M_{ai}^{-1} M_{jb} M_{kc} M_{ld} \alpha_{ijkl} \quad (65)$$

$$\beta'_{abcd} = M_{ai}^{-1} M_{bj} M_{ck}^{-1} M_{dl}^{-1} \beta_{ijkl} \quad (66)$$

$$\mu'_{abcd} = M_{ai}^{-1} M_{bj}^{-1} M_{kc} M_{dl}^{-1} \mu_{ijkl} \quad (67)$$

$$\nu'_{abcd} = M_{ia} M_{jb} M_{ck}^{-1} M_{ld} \nu_{ijkl} \quad (68)$$

Note that since unitary transformations generally preserve the commutation relations, the transformations Eqs.61 are noncanonical and lead to commutation relations that describe different physical behaviour.

The two Heisenberg algebras \mathcal{A}_1 and \mathcal{A}_2 will also serve as examples for fixed background noncommutative geometries in our quantum field theoretical studies.

3 Quantum field theory with minimal uncertainties

In Sec.3.1 a general approach to the path integral formulation of quantum field theories on noncommutative geometric spacetimes is applied. As an example, euclidean ϕ^4 -theory is formulated in Sec.3.2 on the spacetimes \mathcal{A}_1 and \mathcal{A}_2 , using the previously developed Bargmann Fock space techniques. The structure constants of the point-wise multiplication of fields are calculated in Sec.3.3. The Feynman rules are derived in Sec.3.4 and, using their asymptotic behaviour it is shown in Sec.3.5 that, on the spacetimes \mathcal{A}_1 and \mathcal{A}_2 , all graphs of ϕ^4 -theory are regularised.

3.1 Path integral on noncommutative geometric spaces

In the euclidean path integral formulation a field theory is defined through its partition function.

$$Z = N \int_F D\phi e^{-1/\hbar S[\phi]} \quad (69)$$

where N is a normalisation constant and

$$S : F \rightarrow \mathbb{R} \quad S : \phi \rightarrow S[\phi] \quad (70)$$

is a nonlinear action functional from the space F of fields to the real numbers.

The space F of fields $F \subset H$ in a Hilbert space H is a $*$ -representation of the Heisenberg algebra \mathcal{A} generated by elements \mathbf{x}_i and \mathbf{p}_j , ordinarily obeying

$$[\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{i,j} \quad (71)$$

The closure of F under addition insures the translation invariance of the path integral. The \mathbf{p}_i act on fields e.g. in the kinetic action, while the \mathbf{x}_i act on fields e.g. in gauge transformations $\psi \rightarrow \exp(i\alpha(\mathbf{x}))\psi$.

Of course, in quantum field theory the generators \mathbf{x}_i and \mathbf{p}_j of the Heisenberg algebra \mathcal{A} do no longer have the simple quantum mechanical interpretation as observables of positions and momentum, because of the existence of antiparticles. Nevertheless, positions and momenta do not become mere parameters in quantum field theory. It is this Heisenberg algebra \mathcal{A} which is setting the quantum theoretical stage of position and momentum spaces, also in quantum field theory, see also e.g. [47, 48].

Generally, the action functionals S of local field theories can be expressed in terms of the action of \mathcal{A} on fields $\phi \in F$, where F is a $*$ -representation, the scalar product $\mathbf{sp}(\cdot, \cdot)$ in F , and the pointwise multiplication $*$ of fields:

$$* : F \otimes F \rightarrow F \quad (72)$$

Let us consider the example of charged ϕ^4 theory:

$$Z[J] := N \int D\phi e^{\int d^4x \phi^*(\partial_i \partial_i - \mu^2)\phi - \frac{\lambda}{4!}(\phi\phi)^*\phi\phi + \phi^*J + J^*\phi} \quad (73)$$

Here velocities and actions are measured as multiples of c and \hbar . Reintroducing the fundamental constants, together with a unit length l , yields:

$$Z[J] = N \int D\phi D\phi^* e^{\int d^4x \frac{-l^2}{\hbar^2} \phi^*(\mathbf{p}_i \mathbf{p}_i + m^2 c^2) \cdot \phi - \frac{\lambda l^4}{4!} (\phi\phi)^* \phi\phi + \phi^*J + J^*\phi} \quad (74)$$

The choice of l does not affect the theory since it can be absorbed in a finite redefinition of the fields and the coupling constant. It will of course drop out of the Feynman rules.

The Heisenberg algebra \mathcal{A} defined by Eq.71 acts on the fields as

$$\mathbf{x}_i \cdot \phi(x) = x_i \phi(x) \quad \mathbf{p}_j \cdot \phi(x) = -i\hbar \partial / \partial x_j \phi(x), \quad (75)$$

we define a scalar product $\mathbf{sp}(\cdot, \cdot)$ in F

$$\mathbf{sp}(\phi_1, \phi_2) = \int d^4x \phi_1^*(x) \phi_2(x) \quad (76)$$

and the pointwise multiplication $* : F \otimes F \rightarrow F$:

$$(\phi_1 * \phi_2)(x) = \phi_1(x) \phi_2(x) \quad (77)$$

Since we require the space F of fields ϕ that is to be summed over in the path integral to be a $*$ -representation of the commutation relations Eq.71, a suitable specification of F is $F := S_\infty \subset H := L^2$. The domain F is an analytic (F is a $*$ - \mathcal{A} module) and dense domain in the Hilbert space H of square integrable functions.

The pointwise multiplication $*$ equips F with the structure of a non-unital commutative algebra. F is closed under the associative multiplication, while the identity $\phi(x) \equiv 1$ is neither in F nor in H . The commutativity of $*$

$$\forall \phi_1, \phi_2 \in F : \quad \phi_1 * \phi_2 = \phi_2 * \phi_1 \quad (78)$$

is crucial for the description of bosons and can (and will) be preserved on the non-commutative geometries.

The above definitions yield:

$$Z[J] = N \int_F D\phi \, e^{-\frac{l^2}{\hbar^2} \mathbf{sp}(\phi, (\mathbf{p}^2 + m^2 c^2) \cdot \phi) - \frac{\lambda l^4}{4!} \mathbf{sp}(\phi * \phi, \phi * \phi) + \mathbf{sp}(\phi, J) + \mathbf{sp}(J, \phi)} \quad (79)$$

The units are now fully transparent since, through the introduction of l , the abstract fields $\phi \in F$ do not carry units. Their pointwise product $\phi_1 * \phi_2$ does carry units.

Eq.79 provides a formulation of the path integral which is independent of the choice of a Hilbert basis in $F \subset H$. From Eq.79 one obtains Eq.74 by choosing the spectral representation of the position operators \mathbf{x}_i . Equivalently one may choose other Hilbert bases in H , such as e.g. the spectral representation of the momenta in which the Heisenberg algebra \mathcal{A} acts on the fields as $\mathbf{x}_i \cdot \phi(p) = i\hbar \partial / \partial_{p_i} \phi(p)$ and $\mathbf{p}_j \cdot \phi(p) = p_j \phi(p)$ with the scalar product $\mathbf{sp}(\cdot, \cdot)$ in F reading

$$\mathbf{sp}(\phi_1, \phi_2) = \int d^4 p \, \phi_1^*(p) \phi_2(p) \quad (80)$$

and the pointwise multiplication $*$: $F \otimes F \rightarrow F$ taking the form of the convolution product:

$$(\phi_1 * \phi_2)(p) = (2\pi\hbar)^{-2} \int d^4 k \, \phi_1(k) \phi_2(p - k) \quad (81)$$

The form of Eq.79 is not only representation independent. It is crucial that it does also not rely on fixed commutation relations in the Heisenberg algebra \mathcal{A} .

Our approach to the formulation of quantum field theories on noncommutative geometries is therefore to stick to the abstract form of the action functional, as e.g. in Eq.79, while generalising the Heisenberg algebra \mathcal{A} . This means a generalisation of the ‘stage’ of space-time and energy-momentum on which the field theory is built, technically through changes in the action of the operators on fields, the scalar product and in the pointwise product of fields, which are then reflected in the Feynman rules. Note that the scalar products could be written as traces, using $\mathbf{sp}(a, b) = \sum_n \mathbf{sp}(n, b) \mathbf{sp}(a, n) = \mathbf{tr}(|b\rangle\langle a|)$, with $\{|n\rangle\}_n$ being a Hilbert basis in H .

3.2 ϕ^4 -theory on the geometries \mathcal{A}_1 and \mathcal{A}_2

The framework can be applied for the formulation of quantum field theories on generic noncommutative background geometries which may or may not have certain symmetries, similar to the case of curved background geometries. Here, we will use the nontrivial examples of non Lorentz symmetric noncommutative background geometries \mathcal{A}_1 and \mathcal{A}_2 (e.g. for $n = 4$), since they are known to imply minimal uncertainties and since we can conveniently make use of our previous results on the construction of explicit Hilbert space representations.

Note that the quantum mechanics for Lorentz symmetric examples of suitable noncommutative background geometries was studied in [25] and the corresponding field theoretical studies are in progress [26].

Generally, for practical calculations a representation is needed on a domain F of fields in a Hilbert space H and a Hilbert basis to work in. Neither \mathcal{A}_1 nor \mathcal{A}_2 have spectral representations of the \mathbf{x}_i or \mathbf{p}_j , while the Bargmann Fock representations on the domains $F := D_1$ or D_2 , as developed in Sec.2, can again be used. Fields are given as polynomials or power series $\phi(\bar{\eta}_1, \dots, \bar{\eta}_n)$ rather than as functions $\phi(x)$ or $\phi(p)$, with the action of the operators $\mathbf{x}_i, \mathbf{p}_j$ given by Eq.57.

The abstract action functional of Eq.79 is to be expressed, term by term, in the Bargmann Fock representation.

In the case of \mathcal{A}_2 the scalar product of fields reads, from Eq.59:

$$\mathbf{sp}(\phi_1, \phi_2) = \overline{\phi_1(\bar{\eta})} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \phi_2(\bar{\eta})|_0 \quad (82)$$

Here and in the following we sum over repeated indices and $|_0$ stands for 'all differentiations evaluated at zero'.

The source terms are scalar products:

$$\mathbf{sp}(\phi, J) = \overline{\phi(\bar{\eta})} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} J(\bar{\eta})|_0 \quad (83)$$

$$\mathbf{sp}(J, \phi) = \overline{J(\bar{\eta})} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \phi(\bar{\eta})|_0 \quad (84)$$

From Eq.79 the free part of the action functional is the scalar product of the field ϕ with the field $Q \cdot \phi$:

$$S_0[\phi] = \mathbf{sp}(\phi, Q \cdot \phi) \quad (85)$$

where

$$Q := \frac{l^2}{\hbar^2} (\mathbf{p}_i \mathbf{p}_i + m^2 c^2) \quad (86)$$

which acts on Bargmann Fock space as:

$$\frac{l^2}{\hbar^2} (\mathbf{p}_i \mathbf{p}_i + m^2 c^2) \cdot \phi(\bar{\eta}) = \frac{l^2}{\hbar^2} \left(- \sum_{i=1}^4 K_i^2 (\bar{\eta}_i - \partial_{\bar{\eta}_i})^2 + m^2 c^2 \right) \phi(\bar{\eta}) \quad (87)$$

Thus, the free action reads:

$$\begin{aligned} S_0[\phi] &= \frac{l^2}{\hbar^2} \mathbf{sp} \left(\phi, (\mathbf{p}_i \mathbf{p}_i + m^2 c^2) \cdot \phi \right) \\ &= \frac{l^2}{\hbar^2} \overline{\phi(\bar{\eta})} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \left(- \sum_{i=1}^4 K_i^2 (\bar{\eta}_i - \partial_{\bar{\eta}_i})^2 + m^2 c^2 \right) \phi(\bar{\eta})|_0 \end{aligned} \quad (88)$$

The interaction term is the scalar product of the field $\phi * \phi$ with itself, it thus reads in Bargmann Fock space:

$$\begin{aligned} S_{int}[\phi] &= \frac{\lambda l^4}{4!} \mathbf{sp}(\phi * \phi, \phi * \phi) \\ &= \frac{\lambda l^4}{4!} \overline{(\phi * \phi)(\bar{\eta})} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} (\phi * \phi)(\bar{\eta})|_0 \end{aligned} \quad (89)$$

These are the expressions for \mathcal{A}_2 . In the case of the geometry \mathcal{A}_1 the exponential is replaced by the product of exponentials

$$e^{\sum_{i=1}^4 \partial_{\eta_i} \partial_{\bar{\eta}_i}} \rightarrow \prod_{i=1}^4 e^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \quad (90)$$

while the case of ordinary geometry is of course recovered for q or all $q_i \rightarrow 0$.

Recall that the $\bar{\eta}_i$ have two multiplicative structures, related to the Heisenberg algebra \mathcal{A} and to the algebra F of fields. Solving Eqs.57 for $\bar{\eta}_i$, the $\bar{\eta}_i$ act as multiplication operators on the fields and can be identified with elements of the Heisenberg algebra, thus, in the generalised case, reflecting its noncommutativity. On the other hand the fields $\phi(\bar{\eta})$ are commutatively multiplied pointwise, through ‘ $*$ ’ for the description of local interaction.

The structure constants $C_{\vec{r}, \vec{s}, \vec{t}}$ (here and in the following index ‘vectors’ \vec{r} take values $\vec{r} \in \mathbb{N}^4$)

$$C_{\vec{r}, \vec{s}, \vec{t}} := \mathbf{sp}(\bar{\eta}_1^{r_1} \cdot \dots \cdot \bar{\eta}_4^{r_4}, \bar{\eta}_1^{s_1} \cdot \dots \cdot \bar{\eta}_4^{s_4} * \bar{\eta}_1^{t_1} \cdot \dots \cdot \bar{\eta}_4^{t_4}) \quad (91)$$

will be needed explicitly.

3.3 Pointwise multiplication

On ordinary geometry the pointwise multiplication $*$ transforms into momentum space as the well known convolution product:

$$(\phi * \phi)(x) = \phi(x)\phi(x) \quad (92)$$

$$(\phi * \phi)(p) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{+\infty} dk \phi_p(k)\phi_p(p-k) \quad (93)$$

In order to obtain the convolution product formula, two arbitrary functions on momentum space are unitarily (Fourier-) transformed into position space, multiplied pointwise, and the resulting function is unitarily (Fourier-) transformed back into momentum space, yielding Eq.93.

Analogously the unitary equivalence of the Bargmann Fock- with the position space representation determines the pointwise multiplication in Bargmann Fock space and the $C_{\vec{r}, \vec{s}, \vec{t}}$ uniquely:

The matrix elements of the unitary transformation to the spectral representation of \mathbf{x} are

$$\langle x | \bar{\eta}^n \rangle = \sqrt{n!} (2\pi L^2)^{-1/4} (x/2L - L\partial_x)^n e^{-\frac{1}{4}(\frac{x}{L})^2} \quad (94)$$

i.e., up to a factor, the Hermite functions. The use of Eq.94 for the transformation of $*$ from position space into Bargmann Fock space is however rather inconvenient. Starting from known expressions, more practical formulas can be developed.

As has been known since [49] (see also [50] and references therein), fields $\phi(x)$ given in

the position representation are transformed into the Bargmann Fock representation by

$$\phi(\bar{\eta}) = (2\pi L^2)^{-1/4} \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}\bar{\eta}^2 + \bar{\eta}\frac{x}{L} - \frac{1}{4}(\frac{x}{L})^2} \phi(x) \quad (95)$$

with the inverse:

$$\phi(x) = (8\pi^3 L^2)^{-1/4} \int_{-i\infty}^{+i\infty} d\bar{\eta} e^{\frac{1}{2}\bar{\eta}^2 - \bar{\eta}\frac{x}{L} + \frac{1}{4}(\frac{x}{L})^2} \phi(\bar{\eta}) \quad (96)$$

To see this, note that the Bargmann Fock function $\phi(\bar{\eta}) := 1$ is mapped onto

$$\phi(x) = (2\pi L^2)^{-1/4} e^{-1/4 (x/L)^2} \quad (97)$$

and vice versa. The induction is then completed by showing that multiplying the Bargmann Fock function with $\bar{\eta}$ amounts to the action of $(x/L - 2L\partial_x)/2$ on the field in position space.

These formulas, connecting the position space with the Bargmann Fock space, are analogues of the Fourier transformation formulas connecting the position space with the momentum space. Similar formulas connect Bargmann Fock space directly to momentum space:

$$\phi(\bar{\eta}) = \left(\frac{2L^2}{\pi\hbar^2}\right)^{1/4} \int_{-\infty}^{+\infty} dp e^{\frac{1}{2}\bar{\eta}^2 + 2i\bar{\eta}\frac{Lp}{\hbar} - \left(\frac{Lp}{\hbar}\right)^2} \phi_p(p) \quad (98)$$

with the inverse:

$$\phi_p(p) = \left(\frac{L^2}{2\pi^3\hbar^2}\right)^{1/4} \int_{-\infty}^{+\infty} d\bar{\eta} e^{-\frac{1}{2}\bar{\eta}^2 - 2i\bar{\eta}\frac{Lp}{\hbar} + \left(\frac{Lp}{\hbar}\right)^2} \phi(\bar{\eta}) \quad (99)$$

For the proof, note that $\mathbf{sp}(p, \phi(\bar{\eta}) = 1) = \left(\frac{2L^2}{\pi\hbar^2}\right)^{1/4} e^{-\left(\frac{Lp}{\hbar}\right)^2}$.

Let us remark that from Eqs.95,96 immediately follows that the transformation

$$\tilde{f}(y) := \int_{-\infty}^{+\infty} dx e^{-\frac{1}{L^2}(x-y)^2} f(x) \quad (100)$$

which yields a ‘Gaussian-diluted’ function has an inverse:

$$f(x) = \frac{1}{\pi L^2} \int_{-\infty}^{+\infty} dy e^{\frac{1}{L^2}(x-iy)^2} \tilde{f}(iy) \quad (101)$$

The \mathbf{x} -eigenvector with eigenvalue x_0 , i.e. in position space the ‘ δ - function’ at x_0 , has the Bargmann Fock representation $\phi_{(x_0)}(\bar{\eta})$ (using Eq.95):

$$\phi_{(x_0)}(\bar{\eta}) = (2\pi L^2)^{-1/4} e^{-\frac{\bar{\eta}^2}{2} + \bar{\eta}\frac{x_0}{L} - \frac{1}{4}(\frac{x_0}{L})^2} \quad (102)$$

The scalar product of an arbitrary $\phi(\bar{\eta})$ with $\phi_{(x_0)}(\bar{\eta})$ yields another formula for the transformation from Bargmann Fock to position space, using Eq.17:

$$\phi(x) = \frac{(2\pi L^2)^{-1/4}}{2\pi i} \int d\eta d\bar{\eta} e^{-\bar{\eta}\eta - \frac{\eta^2}{2} + \eta \frac{x}{L} - \frac{1}{4}(\frac{x}{L})^2} \phi(\bar{\eta}) \quad (103)$$

Similarly, the use the algebraic form Eq.18 of the scalar product yields

$$\phi(x) = (2\pi L^2)^{-1/4} e^{-\frac{\eta^2}{2} + \eta \frac{x}{L} - \frac{1}{4}(\frac{x}{L})^2} e^{\partial_{\eta} \partial_{\bar{\eta}}} \phi(\bar{\eta})|_{\eta=0=\bar{\eta}} \quad (104)$$

and thus:

$$\phi(x) = (2\pi L^2)^{-1/4} e^{-\frac{1}{2}\partial_{\bar{\eta}}^2 + \frac{x}{L}\partial_{\bar{\eta}} - \frac{1}{4}(\frac{x}{L})^2} \phi(\bar{\eta})|_{\bar{\eta}=0} \quad (105)$$

This new transformation formula no longer involves integrations and can be evaluated algebraically, using the Leibniz rule only.

The pointwise multiplication on Bargmann Fock space is now calculated by unitarily transforming two arbitrary Bargmann Fock functions into position space, using the new formula Eq.105, multiplying pointwise, and unitarily transforming the resulting function back into Bargmann Fock space, using Eq.95, to obtain:

$$\begin{aligned} (\phi_1 * \phi_2)(\bar{\eta}) &= (2\pi L^2)^{-\frac{3}{4}} \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}(\bar{\eta}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{x}{L}(\bar{\eta} + \partial_{\bar{\eta}'} + \partial_{\bar{\eta}''}) - \frac{3}{4}(x/L)^2} \phi_1(\bar{\eta}') \phi_2(\bar{\eta}'')|_0 \\ &= \left(\frac{2}{9\pi L^2}\right)^{\frac{1}{4}} e^{\frac{1}{3}(\bar{\eta} + \partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})^2 - \frac{1}{2}(\bar{\eta}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2)} \phi_1(\bar{\eta}') \phi_2(\bar{\eta}'')|_0 \end{aligned} \quad (106)$$

This is the convolution product formula for Bargmann Fock space. It allows to calculate the C_{rst} :

$$\begin{aligned} C_{rst} &= \mathbf{sp}(\bar{\eta}^r, \bar{\eta}^s * \bar{\eta}^t) \\ &= \eta^r e^{\partial_{\eta} \partial_{\bar{\eta}}} \left(\frac{2}{9\pi L^2}\right)^{\frac{1}{4}} e^{\frac{1}{3}(\bar{\eta} + \partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})^2 - \frac{1}{2}(\bar{\eta}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2)} \bar{\eta}'^s \bar{\eta}''^t|_0 \\ &= \left(\frac{2}{9\pi L^2}\right)^{\frac{1}{4}} e^{\frac{1}{3}(\partial_{\bar{\eta}} + \partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})^2 - \frac{1}{2}(\partial_{\bar{\eta}}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2)} \bar{\eta}^r \bar{\eta}'^s \bar{\eta}''^t|_0 \end{aligned} \quad (107)$$

Using

$$\partial_x^r e^{ax+bx^2}|_{x=0} = \sum_{s \leq r/2} \frac{r!}{s!(r-2s)!} a^{r-2s} b^s \quad (108)$$

we evaluate

$$\begin{aligned} &e^{\frac{-1}{6}(\partial_{\bar{\eta}}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}} \partial_{\bar{\eta}'} + \partial_{\bar{\eta}} \partial_{\bar{\eta}''} + \partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \bar{\eta}^r \bar{\eta}'^s \bar{\eta}''^t|_0 \\ &= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \partial_{\bar{\eta}}^r e^{\frac{-1}{6}\bar{\eta}^2 + \frac{2}{3}\bar{\eta}(\partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})} \bar{\eta}'^s \bar{\eta}''^t|_0 \\ &= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \sum_u \frac{r!}{u!(r-2u)!} \left(\frac{2}{3}(\partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})\right)^{r-2u} \left(\frac{-1}{6}\right)^u \bar{\eta}'^s \bar{\eta}''^t|_0 \end{aligned}$$

$$\begin{aligned}
&= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \sum_u \frac{r! \left(\frac{2}{3}\right)^{r-2u} \left(\frac{-1}{6}\right)^u}{u!(r-2u)!} \sum_{v=0}^{r-2u} \binom{r-2u}{v} \partial_{\bar{\eta}'}^{r-2u-v} \partial_{\bar{\eta}''}^v \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \sum_{u,v} \frac{r! \left(\frac{2}{3}\right)^{r-2u} \left(\frac{-1}{6}\right)^u}{u!(r-2u-v)!v!} \partial_{\bar{\eta}'}^{r-2u-v} \partial_{\bar{\eta}''}^v \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \sum_{u,v} \frac{r!s!t! \left(\frac{2}{3}\right)^{r-2u} \left(\frac{-1}{6}\right)^u \bar{\eta}'^{s-r+2u+v} \bar{\eta}''^{t-v}}{u!(r-2u-v)!v!(s-r+2u+v)!(t-v)!}|_0
\end{aligned}$$

which is, substituting u by $a := r - 2u - v$

$$\begin{aligned}
&= e^{\frac{-1}{6}(\partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\partial_{\bar{\eta}'} \partial_{\bar{\eta}''})} \sum_{a,v} \frac{r!s!t! \left(\frac{2}{3}\right)^{a+v} \left(\frac{-1}{6}\right)^{\frac{r-v-a}{2}}}{\left(\frac{r-v-a}{2}\right)!v!a!(s-a)!(t-v)!} \bar{\eta}'^{s-a} \bar{\eta}''^{t-v}|_0 \\
&= \sum_{a,v} \frac{r!s!t! \left(\frac{2}{3}\right)^{a+v} \left(\frac{-1}{6}\right)^{\frac{r-v-a}{2}}}{\left(\frac{r-v-a}{2}\right)!v!a!(s-a)!(t-v)!} \partial_{\bar{\eta}'}^{s-a} e^{\frac{-1}{6}(\bar{\eta}'^2 + \partial_{\bar{\eta}''}^2) + \frac{2}{3}(\bar{\eta}' \partial_{\bar{\eta}''})} \bar{\eta}''^{t-v}|_0 \\
&= \sum_{a,v,w} \frac{r!s!t! \left(\frac{2}{3}\right)^{a+v} \left(\frac{-1}{6}\right)^{\frac{r-v-a}{2}}}{\left(\frac{r-v-a}{2}\right)!v!a!(s-a)!(t-v)!} \frac{(s-a)! \left(\frac{2}{3}\right)^{s-a-2w} \left(\frac{-1}{6}\right)^w}{(s-a-2w)!w!} \partial_{\bar{\eta}''}^{s-a-2w} e^{\frac{-1}{6}\partial_{\bar{\eta}''}^2} \bar{\eta}''^{t-v}|_0
\end{aligned}$$

and, replacing w by $z := s - a - 2w$

$$\begin{aligned}
&= \sum_{a,v,z} \frac{r!s!t! \left(\frac{2}{3}\right)^{a+v+z} \left(\frac{-1}{6}\right)^{\frac{r-v+s-2a-z}{2}}}{a!v!z! \left(\frac{r-a-v}{2}\right)! \left(\frac{s-a-z}{2}\right)! (t-v-z)!} e^{\frac{-1}{6}\partial_{\bar{\eta}''}^2} \bar{\eta}''^{t-v-z}|_0 \\
&= \sum_{a,v,z} \frac{r!s!t! \left(\frac{2}{3}\right)^{a+v+z} \left(\frac{-1}{6}\right)^{\frac{r-v+s-2a-z}{2}}}{a!v!z! \left(\frac{r-a-v}{2}\right)! \left(\frac{s-a-z}{2}\right)! (t-v-z)!} \left(\frac{-1}{6}\right)^{\frac{t-v-z}{2}} \frac{(t-v-z)!}{\left(\frac{t-v-z}{2}\right)!} \quad (109)
\end{aligned}$$

to obtain eventually:

$$C_{rst} = \left(\frac{2}{9\pi L^2}\right)^{\frac{1}{4}} \sum_{i_1, i_2, i_3} \frac{r! s! t! (-4)^{i_1+i_2+i_3} (-6)^{-\frac{r+s+t}{2}}}{i_1! i_2! i_3! \left(\frac{r-i_2-i_3}{2}\right)! \left(\frac{s-i_1-i_3}{2}\right)! \left(\frac{t-i_1-i_2}{2}\right)!} \quad (110)$$

In the sum over the i_1, i_2, i_3 only those terms contribute for which the arguments of all factorials are positive integers, which is a finite number of terms.

Let us also consider an alternative pointwise multiplication \ast' , which is infrared modified:

$$(\phi \ast' \phi)(x) := \phi(x) \phi(x) e^{\frac{1}{4} \left(\frac{x}{L}\right)^2} \quad (111)$$

In Bargmann Fock space this now takes a simple form without the square of derivatives in the exponential:

$$(\phi \ast' \phi)(\bar{\eta}) = (2\pi L^2)^{-\frac{3}{4}} \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}(\bar{\eta}^2 + \partial_{\bar{\eta}'}^2 + \partial_{\bar{\eta}''}^2) + \frac{x}{L}(\bar{\eta} + \partial_{\bar{\eta}'} + \partial_{\bar{\eta}''}) - \frac{1}{2}(x/L)^2} \phi(\bar{\eta}') \phi(\bar{\eta}'')|_0$$

$$\begin{aligned}
&= (2\pi L^2)^{-\frac{1}{4}} e^{\bar{\eta} \partial_{\bar{\eta}''} + (\bar{\eta} + \partial_{\bar{\eta}''}) \partial_{\bar{\eta}'}} \phi(\bar{\eta}') \phi(\bar{\eta}'')|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} \phi(\bar{\eta} + \partial_{\bar{\eta}'}) \phi(\bar{\eta} + \bar{\eta}')|_0
\end{aligned} \tag{112}$$

Thus

$$\begin{aligned}
C'_{rst} &= \mathbf{sp}(\bar{\eta}^r, \bar{\eta}^s * \bar{\eta}^t) \\
&= \eta^r e^{\partial_{\eta} \partial_{\bar{\eta}}} (2\pi L^2)^{-\frac{1}{4}} e^{\bar{\eta} \partial_{\bar{\eta}''} + (\bar{\eta} + \partial_{\bar{\eta}''}) \partial_{\bar{\eta}'}} \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} e^{\partial_{\bar{\eta}'} \partial_{\bar{\eta}''} + (\partial_{\bar{\eta}'} + \partial_{\bar{\eta}''}) \partial_{\bar{\eta}}} \bar{\eta}^r \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} e^{\partial_{\bar{\eta}'} \partial_{\bar{\eta}''}} (\partial_{\bar{\eta}'} + \partial_{\bar{\eta}''})^r \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} e^{\partial_{\bar{\eta}'} \partial_{\bar{\eta}''}} \sum_{a=0}^r \binom{r}{a} \partial_{\bar{\eta}'}^a \partial_{\bar{\eta}''}^{r-a} \bar{\eta}'^s \bar{\eta}''^t|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} e^{\partial_{\bar{\eta}'} \partial_{\bar{\eta}''}} \binom{r}{a} \frac{s! t!}{(s-a)!(t-r+a)!} \bar{\eta}'^{(s-a)} \bar{\eta}''^{(t-r+a)}|_0 \\
&= (2\pi L^2)^{-\frac{1}{4}} \frac{r! s! t!}{\left(\frac{-r+s+t}{2}\right)! \left(\frac{r-s+t}{2}\right)! \left(\frac{r+s-t}{2}\right)!}
\end{aligned} \tag{113}$$

whenever the arguments of all factorials are positive integers, and zero otherwise. Compare with the C_{rst} which can be put into the form:

$$C_{rst} = \left(\frac{2}{9\pi L^2}\right)^{\frac{1}{4}} \sum_{m_i} \frac{r! s! t! (-4)^{m_1+m_2+m_3} (-6)^{-\frac{r+s+t}{2}}}{\left(\frac{m_1+m_2-m_3}{2}\right)! \left(\frac{m_1-m_2+m_3}{2}\right)! \left(\frac{-m_1+m_2+m_3}{2}\right)! \left(\frac{r-m_1}{2}\right)! \left(\frac{s-m_2}{2}\right)! \left(\frac{t-m_3}{2}\right)!} \tag{114}$$

Recall that, in the position representation, the Bargmann Fock polynomials $\bar{\eta}^m$ have the asymptotic behaviour (Eq.94): $\propto \exp(-(x/2L)^2)$. The pointwise multiplication $*$ of the Hermite functions thus yields a function of asymptotic behaviour $\propto \exp(-2(x/2L)^2)$. The weighted multiplication $*$ ' cancels one of the Gaussian factors and thus keeps the asymptotic behaviour unchanged under the multiplication. Thus F is closed also under $*$ '. While it has a modified (and divergent) infrared behaviour it is also strictly local and is commutative.

3.4 Feynman rules

Generally, given a $*$ -representation F of a possibly generalised Heisenberg algebra \mathcal{A} , together with the structure constants C of the pointwise multiplication in this representation, it is possible to evaluate the action functional for arbitrary fields, and to calculate the Feynman rules.

On the example background geometry \mathcal{A}_2 the fields and sources $\phi, J \in F$ are expanded in the Hilbert basis given by the ordered orthonormal polynomials

$$\phi(\bar{\eta}) = \sum_{s_1, s_2, s_3, s_4=0}^{\infty} \phi_{s_1 s_2 s_3 s_4} \frac{\bar{\eta}_1^{s_1} \bar{\eta}_2^{s_2} \bar{\eta}_3^{s_3} \bar{\eta}_4^{s_4}}{\sqrt{[s_1]_q! [s_2]_q! [s_3]_q! [s_4]_q!}} \tag{115}$$

$$J(\bar{\eta}) = \sum_{s_1, s_2, s_3, s_4=0}^{\infty} J_{s_1 s_2 s_3 s_4} \frac{\bar{\eta}_1^{s_1} \bar{\eta}_2^{s_2} \bar{\eta}_3^{s_3} \bar{\eta}_4^{s_4}}{\sqrt{[s_1]_q! [s_2]_q! [s_3]_q! [s_4]_q!}} \quad (116)$$

so that fields $\phi \in F$ are represented by their coefficient vector $\phi_{\vec{r}} := \phi_{r_1, r_2, r_3, r_4}$ with indices $r_i = 0, 1, 2, \dots, \infty$, ($i = 1, \dots, 4$).

In the case \mathcal{A}_2 the algebra generated by the $\bar{\eta}_i$ is noncommutative and the nontrivial fact that the ordered polynomials still form a Hilbert basis is a consequence of the invariance of the Poincaré series (i.e. of the dimensionalities of the subspaces of polynomials of equal grade), which was one of the key conditions in the derivation of the generalised Bargmann Fock calculus, see [10].

The coefficient matrix of the quadratic operator Q (from Eqs.86,87) in the free action functional is obtained as

$$M_{\vec{r}\vec{s}} := \frac{l^2}{\hbar^2} \frac{\overline{\bar{\eta}_1^{r_1} \bar{\eta}_2^{r_2} \bar{\eta}_3^{r_3} \bar{\eta}_4^{r_4}}}{\sqrt{[r_1]_q! \dots [r_4]_q!}} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \left(m^2 c^2 - \sum_{i=1}^4 K_i^2 (\bar{\eta}_i - \partial_{\bar{\eta}_i})^2 \right) \frac{\bar{\eta}_1^{s_1} \bar{\eta}_2^{s_2} \bar{\eta}_3^{s_3} \bar{\eta}_4^{s_4}}{\sqrt{[s_1]_q! \dots [s_4]_q!}} \Big|_0 \quad (117)$$

while the matrix elements of the interaction term read (from Eq.89)

$$V_{\vec{t}\vec{u}\vec{v}\vec{w}} = \frac{\overline{\bar{\eta}_1^{t_1} \bar{\eta}_2^{t_2} \bar{\eta}_3^{t_3} \bar{\eta}_4^{t_4} * \bar{\eta}_1^{u_1} \bar{\eta}_2^{u_2} \bar{\eta}_3^{u_3} \bar{\eta}_4^{u_4}}}{\sqrt{[t_1]_q! \dots [t_4]_q! [u_1]_q! \dots [u_4]_q!}} e_{1/q}^{\partial_{\eta_i} \partial_{\bar{\eta}_i}} \frac{\bar{\eta}_1^{v_1} \bar{\eta}_2^{v_2} \bar{\eta}_3^{v_3} \bar{\eta}_4^{v_4} * \bar{\eta}_1^{w_1} \bar{\eta}_2^{w_2} \bar{\eta}_3^{w_3} \bar{\eta}_4^{w_4}}{\sqrt{[v_1]_q! \dots [v_4]_q! [w_1]_q! \dots [w_4]_q!}} \Big|_0 \quad (118)$$

and, using Eq.91:

$$V_{\vec{t}\vec{u}\vec{v}\vec{w}} = \sum_{z_1, \dots, z_4=0}^{\infty} \frac{C_{\vec{t}, \vec{u}, \vec{z}} C_{\vec{z}, \vec{v}, \vec{w}}}{\prod_{i=1}^4 [z_i]_q! \sqrt{[t_i]_q! [u_i]_q! [v_i]_q! [w_i]_q!}} \quad (119)$$

The formulas given apply to the case \mathcal{A}_2 . For \mathcal{A}_1 , together with Eq.90, the q 's carry indices q_1, \dots, q_4 .

Note that the path integration can be written as the product of a *countably* infinite number of integrations:

$$\begin{aligned} N \int D\phi(x) D\phi^*(x) e^{-S[\phi(x), \phi^*(x)]} &= N \int D\phi D\bar{\phi} e^{-S[\phi(\bar{\eta}), \bar{\phi}(\bar{\eta})]} \\ &= N \int \prod_{r_1, r_2, r_3, r_4=0}^{\infty} d\phi_{r_1, r_2, r_3, r_4} d\phi_{r_1, r_2, r_3, r_4}^* e^{-S[\phi_{\vec{r}}, \phi_{\vec{r}}^*]} \end{aligned} \quad (120)$$

This discretisation of the infinite number of ordinary integrations which form the path integral is not related to the issue of e.g. ultraviolet regularisation. On ordinary geometry it is merely a result of our choice of representation, which is unitarily equivalent to the conventional representations of the Heisenberg algebra \mathcal{A} . Generally, we are simply making use of the fact that the Hilbert space H is separable, i.e. that H has discrete bases.

The Feynman rules can be derived in the standard way, using the generating functional Eq.79, which now reads:

$$Z[J] = N \int D\phi D\phi^* e^{-\phi_{\vec{r}}^* M_{\vec{r}\vec{s}} \phi_{\vec{s}} - \frac{\lambda L^4}{4!} V_{\vec{t}\vec{u}\vec{v}\vec{w}} \phi_{\vec{t}}^* \phi_{\vec{u}}^* \phi_{\vec{v}} \phi_{\vec{w}} + \phi_{\vec{r}}^* J_{\vec{r}} + J_{\vec{r}}^* \phi_{\vec{r}}} \quad (121)$$

Recall that each index vector denotes four indices, corresponding to the four euclidean dimensions, e.g. $\vec{r} = (r_1, r_2, r_3, r_4)$ where each index is summed over, e.g. $r_2 = 0, 1, 2, \dots \infty$. Pulling the interaction term in front of the integral yields:

$$Z[J] = N e^{-\frac{\lambda L^4}{4!} V_{\vec{r}\vec{s}\vec{t}\vec{u}} \frac{\partial}{\partial J_{\vec{r}}^*} \frac{\partial}{\partial J_{\vec{s}}^*} \frac{\partial}{\partial J_{\vec{t}}^*} \frac{\partial}{\partial J_{\vec{u}}^*}} \int D\phi D\phi^* e^{-\phi_{\vec{r}}^* M_{\vec{r}\vec{s}} \phi_{\vec{s}} + \phi_{\vec{r}}^* J_{\vec{r}} + J_{\vec{r}}^* \phi_{\vec{r}}} \quad (122)$$

In the discrete representation the functional derivatives become ordinary partial derivatives. Rearranging the remaining integrand

$$Z[J] = N e^{-\frac{\lambda L^4}{4!} V_{\vec{r}\vec{s}\vec{t}\vec{u}} \frac{\partial}{\partial J_{\vec{r}}^*} \frac{\partial}{\partial J_{\vec{s}}^*} \frac{\partial}{\partial J_{\vec{t}}^*} \frac{\partial}{\partial J_{\vec{u}}^*}} \int D\phi D\phi^* e^{-(\phi_{\vec{r}}^* - J_{\vec{s}}^* M_{\vec{s}\vec{r}}^{-1}) M_{\vec{r}\vec{t}} (\phi_{\vec{t}} - M_{\vec{t}\vec{u}}^{-1} J_{\vec{u}}) + J_{\vec{r}}^* M_{\vec{r}\vec{s}}^{-1} J_{\vec{s}}} \quad (123)$$

the path integral can be absorbed in the overall constant:

$$Z[J] = N' e^{-\frac{\lambda L^4}{4!} V_{\vec{r}\vec{s}\vec{t}\vec{u}} \frac{\partial}{\partial J_{\vec{r}}^*} \frac{\partial}{\partial J_{\vec{s}}^*} \frac{\partial}{\partial J_{\vec{t}}^*} \frac{\partial}{\partial J_{\vec{u}}^*}} e^{J_{\vec{r}}^* M_{\vec{r}\vec{s}}^{-1} J_{\vec{s}}} \quad (124)$$

The calculation of graphs now involves loop summations rather than loop integrations, with the Feynman rule for the free propagator

$$\Delta_0(\vec{a}, \vec{b}) = M_{\vec{a}\vec{b}}^{-1} \quad (125)$$

and the lowest order vertex:

$$\Gamma_0(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = -\frac{\lambda L^4}{4!} V_{\vec{a}\vec{b}\vec{c}\vec{d}} \quad (126)$$

In graphs, each internal propagator Δ_0 is attached to two legs of a vertex Γ_0 . While the propagator carries a factor of l^{-2} , each leg of the vertex carries a factor l . Thus, as it should be, the length scale l drops out of the calculation.

The Feynman rules could e.g. be applied to the calculation of the first order correction to the propagator, i.e. to the tadpole graph which now reads:

$$\Delta(\vec{a}, \vec{b}) = M_{\vec{a}\vec{b}}^{-1} - \frac{\lambda L^4}{3!} \sum_{\vec{r}, \vec{s}, \vec{t}, \vec{u}} V_{\vec{r}\vec{s}\vec{t}\vec{u}} M_{\vec{u}\vec{a}}^{-1} M_{\vec{t}\vec{r}}^{-1} M_{\vec{b}\vec{s}}^{-1} + \dots \quad (127)$$

where e.g. $\sum_{\vec{r}}$ denotes $\sum_{r_1, r_2, r_3, r_4=0}^{\infty}$. On ordinary geometry the tadpole contribution is divergent, since it reads in momentum space, up to the external legs and a constant,

$$\int d^4p \frac{1}{p_i p_i + m^2 c^2} = \text{quadr. UV divergent} \quad (128)$$

On ordinary geometry, i.e. with the ordinary Heisenberg algebra \mathcal{A} underlying, the Feynman rules in the Bargmann Fock representation are of course equivalent to those in the then existing position and momentum representations, the change of Hilbert basis in F is unitary, its determinant is trivial and no anomalies are introduced. While n -point functions $\Gamma^{(n)}(\vec{x}_1, \dots, \vec{x}_n)$ and $\Gamma^{(n)}(\vec{p}_1, \dots, \vec{p}_n)$ are related by unitary (Fourier-) transformations they can also be transformed unitarily to and from the Bargmann Fock representation $\Gamma^{(n)}(\vec{r}_1, \dots, \vec{r}_n)$, using the transformations given in Eqs.95,96,98,99. On geometries with minimal uncertainties there is still the possibility of unitarily transforming to quasi-position and quasi-momentum representations, see [25] and Sec.3.2.

3.5 Regularisation

The aim now is to investigate whether nonzero minimal uncertainties have the power to regularise the divergencies in ϕ^4 - theory, i.e. whether the loop summations of perturbation theory, such as those in Eq.127, converge on geometries with minimal uncertainties.

For the study of the convergence properties of loop summations the behaviour of the matrix elements of V and M^{-1} for large summation indices needs to be established.

In ϕ^4 - theory on the example geometries $\mathcal{A}, \mathcal{A}_1$ or \mathcal{A}_2 the Feynman rule for the Vertex V is specified by applying the explicit expression Eq.110 for the pointwise multiplication to Eq.119 and Eq.126.

Recall that in the expression Eq.110 for the C_{rst} only those terms contribute to the sum for which the arguments of all factorials are integers (Note also that C_{rst} vanishes if $r + s + t$ is odd²). Thus, for fixed r, s, t , the number of nonzero terms in the sum cannot exceed $r \cdot s \cdot t$ and the C_{rst} can therefore be majorized by:

$$|C_{rst}| < \left(\frac{2}{9\pi L^2} \right)^{\frac{1}{4}} rst r!s!t! 3^{r+s+t} \quad (129)$$

Using $n! < \sqrt{2\pi n} n^n e^{-n} e^{1/12n}$ (from expanding the Gamma function) yields:

$$|C_{rst}| < \left(\frac{2}{9\pi L^2} \right)^{\frac{1}{4}} (2\pi rst)^{3/2} (3/e)^{r+s+t} e^{1/12r+1/12s+1/12t} r^r s^s t^t \quad (130)$$

Splitting off the non-dominant factors

$$k(n) := (2\pi n)^{3/2} (3/e)^n e^{1/12n} \quad (131)$$

yields in 4 dimensions:

$$|C_{\vec{r}, \vec{s}, \vec{t}}| < \frac{2}{9\pi} \prod_{i=1}^4 L_i^{-1/2} k(r_i)k(s_i)k(t_i) r_i^{r_i} s_i^{s_i} t_i^{t_i} \quad (132)$$

²In position and momentum space, this is the integral over the product of three odd Hermite functions.

The denominator in Eq.119 reflects changes arising with the modified geometry. The estimate

$$\begin{aligned}
[n]_q! &= \prod_{a=1}^n [a]_q = \prod_{a=1}^n \sum_{b=0}^{a-1} q^{2b} \\
&= (1+q^2)(1+q^2+q^4) \cdot \dots \cdot (1+\dots+q^{2(n-1)}) \\
&> q^{2(1+2+\dots+n-1)} = q^{n^2-n}
\end{aligned} \tag{133}$$

yields for the geometry \mathcal{A}_2 :

$$|V_{\vec{t}\vec{u}\vec{v}\vec{w}}| < \frac{4}{81\pi^2} \sum_{z_1, \dots, z_4} \prod_{i=1}^4 L_i^{-1} \frac{k^2(z_i) z_i^{2z_i} k(t_i) k(u_i) k(v_i) k(w_i) t_i^{t_i} u_i^{u_i} v_i^{v_i} w_i^{w_i}}{q^{z_i^2 - z_i + (t_i^2 - t_i + u_i^2 - u_i + v_i^2 - v_i + w_i^2 - w_i)/2}} \tag{134}$$

The same majorisation holds in the case \mathcal{A}_1 (where there are four q_i rather than a q), then defining $q := \min(q_1, \dots, q_4)$.

The sums are convergent and can be absorbed in a finite dimensionless constant $K^4(q)$:

$$K(q) := \sum_{z=0}^{\infty} k(z) z^{2z} q^{-z^2+z} \tag{135}$$

to yield for the elementary vertex, using Eq.126:

$$|\Gamma_0(\vec{t}, \vec{u}, \vec{v}, \vec{w})| < \frac{\lambda l^4}{4!} \frac{4}{81\pi^2} K^4(q) \prod_{i=1}^4 L_i^{-1} \frac{k(t_i) k(u_i) k(v_i) k(w_i) t_i^{t_i} u_i^{u_i} v_i^{v_i} w_i^{w_i}}{q^{(t_i^2 - t_i + u_i^2 - u_i + v_i^2 - v_i + w_i^2 - w_i)/2}} \tag{136}$$

Note that $\Gamma_{\vec{t}\vec{u}\vec{v}\vec{w}}$ is now, i.e. for $q > 1$, highly suppressed for large indices. It remains to investigate the high index behaviour of the Feynman rule $\Delta_0(\vec{a}, \vec{b})$ of the free propagator.

We remark that, e.g. on \mathcal{A}_1 , the modified inverse propagator (summed over i)

$$Q' := \frac{l^2}{\hbar^2} \left(\mathbf{p}_i \mathbf{p}_i + m^2 c^2 + \left(\frac{\Delta p_{0i}}{\Delta x_{0i}} \right)^2 \mathbf{x}_i \mathbf{x}_i \right) \tag{137}$$

is self-adjoint and reads in the Bargmann Fock representation:

$$Q' \cdot \phi(\vec{\eta}) = \left(\sum_{i=1}^4 \left(\frac{(q_i^2 + 1)^3 l^2}{8L_i^2} \bar{\eta}_i \partial_{\bar{\eta}_i} + \frac{(q_i^2 + 1)^2 l^2}{8L_i^2} \right) + \frac{m^2 c^2 l^2}{\hbar^2} \right) \cdot \phi(\vec{\eta}) \tag{138}$$

Due to

$$\sum_{i=1}^4 \bar{\eta}_i \partial_{\bar{\eta}_i} \cdot (\bar{\eta}_1^{r_1} \dots \bar{\eta}_4^{r_4}) = \sum_{i=1}^4 [r_i]_{q_i} (\bar{\eta}_1^{r_1} \dots \bar{\eta}_4^{r_4}) \tag{139}$$

it is diagonal, yielding the propagator

$$\Delta'_0(\vec{r}, \vec{s}) = \left(\sum_{i=1}^4 \left(\frac{(q_i^2 + 1)^3 l^2}{8L_i^2} [r_i]_{q_i} + \frac{(q_i^2 + 1)^2 l^2}{8L_i^2} \right) + \frac{m^2 c^2 l^2}{\hbar^2} \right)^{-1} \delta_{\vec{r}, \vec{s}} \quad (140)$$

Its nonzero matrix elements rapidly decrease for large indices, due to the exponential behaviour of $[n]_q = (q^{2n} - 1)/(q^2 - 1)$. An analogous calculation is possible on \mathcal{A}_2 . Recall that the propagator $\Delta'_0(\vec{r}, \vec{s})$ approximates $\Delta_0(\vec{r}, \vec{s})$ for $\Delta p_{0i} \rightarrow 0$, i.e. for vanishing minimal uncertainties in momentum and that it should not differ from $\Delta_0(\vec{r}, \vec{s})$ in the ultraviolet.

Nevertheless, an explicit majorisation of the matrix elements of the true propagator $\Delta_0(\vec{r}, \vec{s})$ is needed. Since the Bargmann Fock representation of Q , i.e. M , is nondiagonal, the explicit calculation of $\Delta_0(\vec{r}, \vec{s})$ is rather involved, see Eqs.87,117 and 125. We can however obtain a majorisation of its crucial high-index behaviour:

On F , which is analytic, the operator Q is symmetric and positive definite, thus allowing a canonical, lower bound preserving self-adjoint extension. This so-called Friedrich extension, see e.g.[51], has a self-adjoint and bounded inverse Q^{-1} , defined on the entire Hilbert space, as has every positive definite self-adjoint operator.

It is crucial that, since Q^{-1} is bounded $C(q) := \|Q^{-1}\| < \infty$, also its matrix elements $\Delta_0(\vec{r}, \vec{s})$ are bounded. This follows immediately from the Cauchy Schwarz inequality and yields the majorisation:

$$|\Delta_0(\vec{r}, \vec{s})| \leq C(q) \quad \forall \vec{r}, \vec{s} \in \mathbb{N}^4 \quad (141)$$

In fact, the lower bound of Q on F is now positive even in the absence of a mass term, because, on \mathcal{A}_1 and \mathcal{A}_2

$$\text{sp} \left(\phi, \sum_i \mathbf{p}_i \mathbf{p}_i \cdot \phi \right) \geq \|\phi\|^2 \sum_i (\Delta p_{i0})^2 \quad \forall \phi \in F \quad (142)$$

i.e. technically through what in the language of quantum mechanics is the existence of minimal uncertainties in momentum Δp_{i0} (from Eq.27 and [13]), on these geometries. Since the Friedrich extension preserves the lower bound we obtain self-adjoint and in particular also bounded i.e. infrared regular propagators also in the massless case. More general studies on propagators and infrared regularisation are in progress.

The strong suppression of the matrix elements of the vertex for high indices, together with the boundedness of $\Delta_0(\vec{r}, \vec{s})$ suffices to prove the finiteness of all graphs in ϕ^4 -theory on the geometries \mathcal{A}_1 and \mathcal{A}_2 :

Connected graphs $G^{(n)}(r_1, \dots, r_n)$, consisting of loop summations over n_p free propagators Δ_0 and n_v vertices Γ_0 can be majorised by

$$|G^{(n)}(r_1, \dots, r_n)| < C^{n_p}(q) \left(\frac{l}{3} \sqrt{\frac{\sqrt{\lambda}}{\sqrt{6\pi}}} K(q) \sum_{m=0}^{\infty} k(m) m^m q^{(-m^2+m)/2} \right)^{4n_v} \prod_{i=1}^4 L_i^{-n_v} \quad (143)$$

which is convergent due to the summability of the sequence $s_n := m^m q^{-m^2}$, as is readily checked by the quotient criterion.

Let us recall that the constants $C_{\vec{r},\vec{s},\vec{t}}$ were calculated on the ordinary geometry \mathcal{A} and have been kept invariant while switching on the generalised geometry (i.e. for $q > 1$ or the $q_i > 1$). The noncommutative geometry entered into the Feynman rules through the changes in the action of the operators on the fields and their scalar product. The modified action of the momentum operators entered into the calculation of the propagator, yielding in particular an obvious infrared regularising effect. The modified scalar product Eq.24 of Bargmann Fock polynomials entered into the Vertex, regularising the local interaction.

However, to stick to the $C_{\vec{r},\vec{s},\vec{t}}$ of ordinary geometry, as we did, is only a minimal choice. For generalised Heisenberg algebras which imply minimal uncertainties, such as our \mathcal{A}_1 and \mathcal{A}_2 , there is no unitarily equivalent position space representation of the commutation relations, which would uniquely fix the $C_{\vec{r},\vec{s},\vec{t}}$ of the pointwise multiplication.

We showed the regularity of the field theory without introducing any nonlocality by hand. But in fact, on noncommutative geometric spaces implying minimal uncertainties, the $C_{\vec{r},\vec{s},\vec{t}}$ could be modified by hand to some extent, introducing an apparent regularising nonlocality, without spoiling observational locality. This is because structure constants which would imply a slight nonlocality of the interaction on ordinary geometry are now to be considered observationally local as long as the nonlocality introduced is not larger than the scale of the minimal uncertainty inherent in the underlying geometry, i.e. as long as interaction cannot lead to an observable nonlocality. This issue needs a further careful investigation which will imply the use of maximal localisation states, see [25, 44]. We remark that, as is not difficult to check, regularisation on \mathcal{A}_1 and \mathcal{A}_2 can be proven along the same lines also for the infrared modified pointwise multiplication \ast' which we mentioned in Sec.3.3.

4 Summary and Outlook

In Sec.2 we reviewed and generalised the results of Refs.[10]-[13] in which is studied the quantum mechanics on noncommutative geometric spaces that imply nonvanishing minimal uncertainties in positions and momenta. Technically, the position and momentum operators are symmetric but no longer essentially self-adjoint, a fact that is crucial in the presence of minimal uncertainties, although it is complicating the construction of \ast -representations of the Heisenberg algebra. Physically, the approach leads to a modified behaviour at very small and at very large scales, which can be motivated to arise from gravity and is coinciding with results of string theory.

In Sec.3 we continued the euclidean field theoretical studies of [14, 15]. For two examples of noncommutative geometries that imply minimal uncertainties we worked out the Feynman rules of charged ϕ^4 - theory and were now able to prove the finiteness of

all graphs. The results show, at least in the example of ϕ^4 -theory, that if gravity or string theory effects induce minimal uncertainties, with e.g. Δx_0 of the order of the Planck length, this could indeed provide a natural regularisation of field theories.

Further studies in the context of ultraviolet regularisation and microcausality will use the maximal localisation states to study the locality properties of generalised point-wise multiplications. The properties of maximal localisation states [25, 44] acquire interesting new features in the general n -dimensional situation where the minimal uncertainty gap in the space of the Δx_i and Δp_j can have a complicated structure. This is being analysed first for the simpler case without minimal uncertainties in momenta [26].

We remark that corrections to the commutation relations can imply that e.g. the \mathbf{p}_i then generate nonlinear transformations of the coordinates, which under certain conditions can be interpreted as the translation of normal coordinate frames on a curved space, see [14, 16]. Further studies on this ‘curvature-noncommutativity duality’ are in progress.

For further studies and practical calculations on noncommutative geometries other than the two classes \mathcal{A}_1 and \mathcal{A}_2 which we have covered so far, it is necessary to construct Hilbert space representations of the corresponding generalised Heisenberg algebras. It is not obvious under which conditions the unitary equivalence of $*$ -representations of the Heisenberg commutation relations (in the sense in which it holds on ordinary geometry) still holds for generalised Heisenberg algebras. It may not hold for some noncommutative geometries in which case the investigation of the above mentioned dual, curved situation should be interesting. One may speculate about a possible relation to horizons or nontrivial topology.

The hope is of course that noncommutative geometric methods could provide new techniques for approaching long outstanding problems in quantum gravity, as they were outlined e.g. in [52]. On the other hand, as discussed in [13, 25], quantum theory on geometries with minimal uncertainties in positions could also provide a suitable framework for an effective description of nonpointlike particles, which could be strings and, changing scale, which could also be compound particles, such as nucleons in situations in which details of their internal structure do not contribute, or e.g. various quasiparticles and collective excitations. Work also in this direction is in progress.

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